Measure of Distance and Similarity for Single Valued Neutrosophic Sets with Application in Multi-attribute Decision Making

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ABSTRACT

A single valued neutrosophic sets (SVNSs) is a particular case of neutrosophic set, which can handle scientific and engineering applications in real world. Distance, similarity, entropy and cross entropy measures of SVNSs are major tool to solve various real world problems. In the paper, an average distance measure based on Hamming and Hausdorff distance is defined for SVNSs. Similarity measure is proposed on the basis of defined averaged distance using the relationship between distance and similarity measures. Further, one more similarity measure is defined and its validity is proved. Then a multi-attribute decision making is also established under single valued neutrosophic environment, in which attribute values are assigned to each course of action by the decision makers/experts and attribute weights are known, whereas the optimal point is defined using the decision matrix using the proposed method. Also, an algorithm is presented to determine the ranking of available course of actions on the basis of values of measure of similarity amongst optimal point and the available course of actions. Lastly, an example is used to demonstrate the application of proposed similarity measures in multi-attribute decision making.
INTRODUCTION

Neutrosophic set developed by Smarandache (1999) is a new evolving instrument for uncertain data processing. It has the potential of being a framework for analysis of uncertain data sets which include big data sets. Neutrosophic sets are generalization of interval valued intuitionistic fuzzy sets. Because of the fuzziness and uncertainty of many practical problems in the real world, it is applicable to a wide range of practical problems. Zadeh (1965) developed fuzzy set theory which is a huge success in various areas involving uncertainty. In fuzzy set theory, non-membership value of an element is defined normally as complement of its membership value from one, but practically it is not so. This situation is dealt by higher order fuzzy sets proposed by Atanassov (1986) and is termed as intuitionistic fuzzy sets (IFSs). It is found to be highly useful in dealing with uncertainty. Many methods have been proposed for measuring the degree of similarity between fuzzy and intuitionistic fuzzy sets which will help in decision making. The cross entropy, distance and similarity for fuzzy and intuitionistic fuzzy sets, neutrosophic sets are also characterized by three functions: truth-membership, indeterminacy-membership and falsity-membership, but neutrosophic sets overcome the aforementioned issues. Therefore, the notion of a neutrosophic set is more general and cannot be dealt by intuitionistic fuzzy sets as sum of three. Thus, it will be difficult to apply these in real scientific and engineering problems. Zadeh (1965) developed fuzzy set theory which is a huge success in various areas involving uncertainty. Thus, when we ask a customer for his/her opinion about a product, it will be difficult to apply these in real scientific and engineering problems. Thus, when we ask a customer for his/her opinion about a product, it will be difficult to apply these in real scientific and engineering problems. Therefore, IFSs are much more flexible and practical than neutrosophic sets and demonstrates its use in multi-attribute decision making, using an illustrative example.

ASIC CONCEPTS AND RELATED WORK

Neutrosophic sets emerged as a tool to deal with uncertain data. It has the potential to become a general framework for uncertainty analysis in data. The neutrosophic set is a part of neutrosophy and generalizes FS, IvFS, IFS, and IvIFS from a philosophical point of view. Smarandache (1999) derived neutrosophic set which is defined as follows:

Definition 1: Let X be a universe of discourse, a neutrosophic set $A$ in $X$ is characterized by a quadruple $\langle x, T(x), I(x), F(x) \rangle$ i.e. $(x, \text{truth-membership function}, \text{indeterminacy-membership function}, \text{falsity-membership function})$, where $x \in X$. The functions $T(x): X \rightarrow [0, 1]$, $I(x): X \rightarrow [0, 1]$, and $F(x): X \rightarrow [0, 1]$. Thus there is no restriction on the sum so $0 \leq \sup T + I + F(x) \leq 3$.

Wang et al. (2011) proposed a subclass of neutrosophic sets termed as single valued neutrosophic sets which are easier to apply to real scientific and engineering problems and is defined as

Definition 1.1: Let $X$ universe of discourse, a single valued neutrosophic set (SVNS) $A$ in $X$ is characterized by quadruple $\langle x, T(x), I(x), F(x) \rangle$ i.e. $(x, \text{truth-membership function}, \text{indeterminacy-membership function}, \text{falsity-membership function})$, where $x \in X$, $T(x), I(x), F(x) \in [0, 1]$ and $0 \leq T(x) + I(x) + F(x) \leq 3$.

Definition 2 (Set operations on single valued neutrosophic sets): Let $A$ and $B$ be two SVNSs defined by quadruple $\langle x, T(x), I(x), F(x) \rangle$ and $\langle x, T'(x), I'(x), F'(x) \rangle$ respectively, where $x \in X$ then set operations are defined as follows:

1. $A \cup B = \{x, T(x) \vee T'(x), I(x) \vee I'(x), F(x) \vee F'(x) / x \in X\}
2. A \cap B = \{x, T(x) \wedge T'(x), I(x) \wedge I'(x), F(x) \wedge F'(x) / x \in X\}
3. $A - \{x, F_\Delta(x), 1 - I_\Delta(x), I_\Delta(x) \} / x \in X\}
4. $A \subseteq B \iff T(x) \leq T'(x), I(x) \geq I'(x)$ and $F(x) \geq F'(x) \forall x \in X$
5. $A = B \iff T(x) = T'(x), I(x) = I'(x)$ and $F(x) = F'(x) \forall x \in X$

Definition 3 (Distance between two single valued neutrosophic sets): For any two SVNSs $A$ and $B$, a real valued function $D: \text{SVNSs} (X) \times \text{SVNSs} (X) \rightarrow [0, 1]$ is termed as a distance measure of SVNSs on $X$, if it satisfies the below mentioned axioms:

1. Distance between any two SVNSs $A$ and $B$ is zero if $A = B$.  

2) Distance measure is symmetrical w.r.t to any two SvNSs $A$ and $B$.

3) For any three SvNSs $A$, $B$ and $C$ such that $A \subseteq B \subseteq C$, we have $D(A, C) = D(A, B) + D(B, C)$.

Distance between FSs was presented by (Kacprzyk, 1997). Its extension was proposed by Atanassov in 1999 as two dimensional distances whereas third parameter hesitancy degree in distance was introduced by Szmidt and Kacprzyk (2000) for intuitionistic fuzzy sets. Yang & Chiclana (2012) proved three dimensional distance consistency over two dimensional distances. Grzegorzewski (2004) and Park et al. (2007) gave distance measure for IvIFSs and IvIFSs respectively. Broumi and Smarandache (2013) presented weighted distance measures for SvNSs based on Hamming and Euclidean Distances given below:

Weighted Hamming Distance:

$$D_{Hw}(A, B) = \frac{1}{n} \sum_{i=1}^{n} w_i |T_i(x_i) - T_i(x_i)| + |I_i(x_i) - I_i(x_i)|$$

where $w_i$ is weight corresponding to $x_i$ such that $\Sigma w_i = 1$. And weighted Euclidean Distance:

$$D_{Ew}(A, B) = \left( \sum_{i=1}^{n} w_i \left( |T_i(x_i) - T_i(x_i)|^2 + |I_i(x_i) - I_i(x_i)|^2 + |F_i(x_i) - F_i(x_i)|^2 \right) \right)^{1/2}$$

where $w_i$ is weight corresponding to $x_i$ such that $\Sigma w_i = 1$.

**Definition 4 (Similarity between SvNSs):** Let $A$ and $B$ be any two SvNSs, a real valued function $S: \text{IvIFSs} \times \text{IvIFSs} \to [0, 1]$ is defined as a measure of similarity for IvIFSs on $X$, if it satisfies following axioms:

1) Measure of similarity between any two IvIFSs is 1 if $A = B$.

2) Measure of similarity is symmetric w.r.t. any two IvIFSs.

3) For any three IvIFSs $A$, $B$ and $C$ such that $A \subseteq B \subseteq C$. We have $S(A, C) = S(A, B) + S(B, C)$.

From axiomatic definition of distance and similarity measures it is clear that $S(A, B) = 1 - D(A, B)$ where $A$ and $B$ are SvNSs, $D$ and $S$ are distance and similarity measure for SvNSs respectively.

Broumi and Smarandache (2013) presented several similarity measures for SvNSs. Ye (2014) presented similarity measures based on equation (2) and (3) as:

$$S_1(A, B) = 1 - \left( \frac{1}{3} \sum_{i=1}^{n} w_i \left\{ \left[ T_i(x_i) - T_i(x_i) \right]^2 + \left[ I_i(x_i) - I_i(x_i) \right]^2 + \left[ F_i(x_i) - F_i(x_i) \right]^2 \right\} \right)^{1/2}$$

and:

$$S_2(A, B) = 1 - \left( \frac{1}{3} \sum_{i=1}^{n} w_i \left\{ \left[ T_i(x_i) - T_i(x_i) \right]^p + \left[ I_i(x_i) - I_i(x_i) \right]^p + \left[ F_i(x_i) - F_i(x_i) \right]^p \right\} \right)^{1/p}$$

Ye also presented one more similarity measure as:

$$S_3(A, B) = 1 - \frac{1}{3} \sum_{i=1}^{n} w_i \left\{ \left[ T_i(x_i) - T_i(x_i) \right]^p + \left[ I_i(x_i) - I_i(x_i) \right]^p + \left[ F_i(x_i) - F_i(x_i) \right]^p \right\}^{1/p}$$

Ye (2015) proposed improved measures of cosine similarity measures for simplified neutrosophic sets including single valued cosine similarity and interval valued neutrosophic cosine similarity. He also introduced corresponding weighted similarity measures and applied it to medical diagnoses. Aydoğdu (2015) introduced two measures of similarity for SvNSs and developed single entropy measure for the same and applied it to neutrosophic multi-criteria decision making. Ye & Smarandache (2016) introduced refined single-valued neutrosophic sets, developed similarity measure of the same and applied it to multi-criteria decision making.

Szmidt and Kacprzyk (2009) constructed Hausdorff distance between Intuitionistic Fuzzy Sets based on the Hamming metric and particularly give importance to the consistency of the metric used and the essence of the Hausdorff distances. Next section deals with new weighted distance and similarity measures based on averaged distance measure based on Hamming and Hausdorff distance defined as:

$$D_{w}(A, B) = \frac{1}{n} \sum_{i=1}^{n} w_i \left\{ \left[ I_i(x_i) - I_i(x_i) \right]^2 + \left[ T_i(x_i) - T_i(x_i) \right]^2 \right\}$$

Distance, similarity measures and optimal point is proposed in the following sections under single valued neutrosophic environment.

**Distance and Similarity Measures for SvNSs:**

This section defines weighted distance and similarity measures based on equation (4) as follows:

**Definition 5 Distance Measure:** Consider two SvNSs $A$ and $B$, represented by quadruple $(x_i, T_i(x), I_i(x), F_i(x))$ and $(x_i, T_i(x), I_i(x), F_i(x))$ respectively in a universe of discourse $X = \{x_1, x_2, x_3, \ldots, x_n\}$. Weighted distance measure $D_w(A, B)$ based on averaged distance measure is defined as follows:

$$D_w(A, B) = \frac{1}{n} \sum_{i=1}^{n} w_i \left\{ \left[ I_i(x_i) - I_i(x_i) \right]^2 + \left[ T_i(x_i) - T_i(x_i) \right]^2 \right\}$$

where $w_i$ is weight corresponding to $x_i$ such that $\Sigma w_i = 1$.

Theorem 1: *Dw(A, B)* is a valid measure of distance between two SvNSs $A$ and $B$.

Proof: Consider two SvNSs $A$ and $B$ represented by quadruple $(x_i, T_i(x), I_i(x), F_i(x))$ and $(x_i, T_i(x), I_i(x), F_i(x))$ respectively in a universe of discourse $X = \{x_1, x_2, x_3, \ldots, x_n\}$. It is easy to see that *Dw(A, B)* [0, 1] satisfies axioms (1) and (2) defined in Definition 3. In order to prove axiom (3), consider three SvNSs $A$, $B$ and $C$ such that $A \subseteq B \subseteq C$. It is easy to see that $D_w(A, B)$ [0, 1] satisfies axioms (1) and (2) defined in Definition 3.
Then
\[ T_S(x_i) \le T_S(x_j) \le T_S(x_k), I_T(x_i) \ge I_T(x_j) \ge I_T(x_k) \text{ and } F_S(x_i) \ge F_S(x_j) \ge F_S(x_k), \]
thus
\[ |T_S(x_j) - T_S(x_i)| \le T_S(x_j) - T_S(x_i) \text{ and } |T_S(x_k) - T_S(x_i)| \le |T_S(x_k) - T_S(x_j)|, \]
\[ |I_T(x_j) - I_T(x_i)| \le |I_T(x_j) - I_T(x_i)| \text{ and } |I_T(x_k) - I_T(x_i)| \le |I_T(x_k) - I_T(x_j)|, \]
\[ |F_S(x_j) - F_S(x_i)| \le |F_S(x_j) - F_S(x_i)| \text{ and } |F_S(x_k) - F_S(x_i)| \le |F_S(x_k) - F_S(x_j)|. \]

And
\[ \frac{|T_S(x_j) - T_S(x_l)|}{3} \le \frac{|T_S(x_l) - T_S(x_i)|}{3} \le \frac{|T_S(x_i) - T_S(x_l)|}{3} \]
\[ \frac{|I_T(x_j) - I_T(x_l)|}{3} \le \frac{|I_T(x_l) - I_T(x_i)|}{3} \le \frac{|I_T(x_i) - I_T(x_l)|}{3} \]
\[ \frac{|F_S(x_j) - F_S(x_l)|}{3} \le \frac{|F_S(x_l) - F_S(x_i)|}{3} \le \frac{|F_S(x_i) - F_S(x_l)|}{3} \]

Also,
\[ \frac{|T_S(x_j) - T_S(x_l)|}{3} \le \frac{|T_S(x_l) - T_S(x_i)|}{3} \le \frac{|T_S(x_i) - T_S(x_l)|}{3} \]
\[ \frac{|I_T(x_j) - I_T(x_l)|}{3} \le \frac{|I_T(x_l) - I_T(x_i)|}{3} \le \frac{|I_T(x_i) - I_T(x_l)|}{3} \]
\[ \frac{|F_S(x_j) - F_S(x_l)|}{3} \le \frac{|F_S(x_l) - F_S(x_i)|}{3} \le \frac{|F_S(x_i) - F_S(x_l)|}{3} \]

Thus \( D_s(A, B) \) and \( D_s(B, C) \) does \( D_s(A, C) \). Hence \( D_s(A, B) \) is a valid measure of distance.

According to the relationship between the distance and similarity measures, similarity measure corresponding to \( D_s(A, B) \) is derived as follows

\[ S_s(A, B) = 1 - \frac{1}{w_i} \sum_{i} w_i \cdot \frac{|T_S(x_j) - T_S(x_i)|}{3} + \frac{|I_T(x_j) - I_T(x_i)|}{3} + \frac{|F_S(x_j) - F_S(x_i)|}{3} \]

where \( w_i \) is weight corresponding to \( x_i \) such that \( \sum_{i} w_i = 1 \).

Obviously, we can easily prove that \( S_s(A, B) \) satisfies the axioms mentioned in Definition 4. Furthermore, we can also propose another measure of similarity for SVNSs

\[ S(A, B) = \frac{1}{w_i} \sum_{i} w_i \cdot \frac{|T_S(x_j) - T_S(x_i)| + |I_T(x_j) - I_T(x_i)| + |F_S(x_j) - F_S(x_i)|}{3} \]

Theorem 2: \( S(A, B) \) is a valid measure of similarity between two SVNSs \( A \) and \( B \).

Proof: Consider two SVNSs \( A \) and \( B \) represented by quadruple \( (x_i, T_S(x_i), I_T(x_i), F_S(x_i)) \) and \( (x_j, T_S(x_j), I_T(x_j), F_S(x_j)) \) respectively in a universe of discourse \( = \{x_1, x_2, \ldots, x_n\} \). It is easy to see that \( S(A, B) \in [0, 1] \) satisfies axioms (1) and (2) defined in Definition 4.

In order to prove axiom (3), assume that \( A \prec B \prec C \). Then \( T_S(x_j) \le T_S(x_i) \le T_S(x_k), I_T(x_j) \ge I_T(x_i) \ge I_T(x_k) \) and \( F_S(x_j) \ge F_S(x_i) \ge F_S(x_k) \). Thus

\[ T_S(x_i) + I_T(x_i) + F_S(x_i) = T_S(x_j) + I_T(x_j) + F_S(x_j) \]

Also,

\[ T_S(x_i) + I_T(x_i) + F_S(x_i) = T_S(x_j) + I_T(x_j) + F_S(x_j) \]

And

\[ T_S(x_i) + I_T(x_i) + F_S(x_i) = T_S(x_j) + I_T(x_j) + F_S(x_j) \]

Adding equations (8), (9) and (10)

\[ T_S(x_i) + I_T(x_i) + F_S(x_i) = T_S(x_j) + I_T(x_j) + F_S(x_j) \]

Corollary: Weighted similarity measure corresponding to equation (7) can be defined as

\[ S^w(A, B) = \frac{1}{w_i} \sum_{i} w_i \cdot \frac{|T_S(x_j) - T_S(x_i)| + |I_T(x_j) - I_T(x_i)| + |F_S(x_j) - F_S(x_i)|}{3} \]

Next, section defines an optimal point and presents an algorithm that helps to solve multi-attribute decision making problems.

Application of SVNSs Similarity Measures to Multi-attribute Decision Making

This section proposes definition of optimal point and uses proposed similarity measures to draw inferences in multi attribute decision making under single valued neutrosophic environments.

Multi-attribute decision making in an organization involves various attributes along with various decision takers. Recently, it has become more complex. To take any decision, a manager needs to have relevant information and decent analytical skills. In some practical situations the course of action involves incomplete and indeterminate information, which is stated in terms of SVNSs. Similarity measures can be used as a tool to identify best course of action by determining the similarity between each course of action and ideal decision criteria/ideal point. Ideal point does not exist in reality. In order to identify
the substitute of ideal point we have defined optimal point as follows:

\[ A_i = (x_i, \max[T(x_i)], \max[I(x_i)], \min[F(x_i)]), \forall i \]  \hspace{1cm} (18)

The best course of action can be identified on the basis of similarity values between optimal point and available course of actions. Larger the value of similarity more closer it is to the optimal point.

Let us consider a multi-attribute decision making problem involving a set of options \( P = \{P_1, P_2, \ldots, P_m\} \) to be considered on the basis of attributes \( C = \{C_1, C_2, \ldots, C_n\} \).

Assume that the weight of an attribute \( C_j \) \((j = 1, 2, \ldots, n)\), entered by the decision-maker, is \( w_j \in [0, 1] \) and \( \sum_{i=1}^{n} w_i = 1 \). Corresponding to each option \( P_i \) \((i = 1, \ldots, m)\) and attribute \( C_j \), the values of three function \( T_A(x_i), I_A(x_i), \) and \( F_A(x_i) \) denoted by single valued neutrosophic value \( d_i(x_j) \), which is derived from evaluation of each course of action based on each attribute. To identify best course of action, similarity between each course of action and identified optimal course of action is calculated. Higher the value of similarity measure closer it is with the optimal value. Course of action with highest value is identified as best course of action.

Multi-attribute decision process can be summarized as follows:

Step 1: Identify weight corresponding to each attribute.

Step 2: Formulate decision matrix corresponding to each attribute provided by the decision maker.

Step 3: Identify optimal point using equation (18) using decision matrix obtained in step 2.

Step 4: Calculate the similarity value \( S(P_i, A_i) \) or \( S(P_i, A_j) \) or \( S(P_i, A) \) by using equation (6) or (7) or (17).

Step 5: Rank and identify the best alternatives on the basis weighted similarity measure value.

Next subsection explains the procedure of application of similarity measure in multi-attribute decision making using a numerical example.

ILLUSTRATIVE EXAMPLE

Here multi-attribute decision making problem is adopted from Ye (2014) to demonstrate the procedure of multi-attribute decision making. Consider four suppliers \( P = \{P_1, P_2, P_3, P_4\} \) which are concerned with a manufacturing company, that wants to select the best global supplier according to the core competencies of suppliers. The core competencies of suppliers are evaluated on the basis of four attributes: (i) \( C_1 \) is the level of technology innovation; (ii) \( C_2 \) is the control ability of the flow; (iii) \( C_3 \) is the ability of management; (iv) \( C_4 \) is the level of service. Then, the weight vector for the four criteria is \( w = (0.3, 0.25, 0.25, 0.2)^T \). The decision matrix of the suppliers is made according to the four evaluating criteria. Therefore, the single valued neutrosophic decision matrix of the suppliers is as follows:

\[
\begin{align*}
A_1 & = \{(0.5,0.1,0.3), (0.5,0.1,0.4), (0.7,0.1,0.2), (0.3,0.2,0.1)\}, \\
A_2 & = \{(0.4,0.2,0.3), (0.5,0.2,0.4), (0.9,0.0,0.1), (0.5,0.3,0.2)\}, \\
A_3 & = \{(0.4,0.3,0.1), (0.5,0.1,0.3), (0.5,0.0,0.4), (0.6,0.2,0.3)\}, \\
A_4 & = \{(0.6,0.1,0.2), (0.2,0.2,0.5), (0.4,0.3,0.2), (0.7,0.2,0.4)\}.
\end{align*}
\]

To identify the most desirable supplier, we calculate the similarity of each supplier with optimal identified values.

Optimal solution is identified using equation (18) as follows:

\[
A_1 = \{(0.6,0.3,0.1), (0.5,0.2,0.3), (0.9,0.3,0.1), (0.7,0.3,0.1)\}
\]

It can be easily observed that the cross entropy measures proposed by Ye (2014) and Ye (2016) provide ranking \( A_1, A_2, A_3, A_4 \) and \( A_2, A_1, A_3, A_4 \) respectively to the suppliers. But Ye (2014) and Ye (2016) had taken optimal solution as \( (1,0,0) \) which is quite unrealistic. So, the ranking provided by the proposed similarity measure is more reasonable as the optimal solution considered in this paper is more realistic.

Comparison with some Existing Measures of Similarity

The cross entropy measures considered in this paper is more realistic.

- Ye (2014) had taken optimal solution as \( (1,0,0) \) which is quite unrealistic. So, the ranking provided by the proposed similarity measure is more reasonable as the optimal solution considered in this paper is more realistic.

- Ye (2016) had taken optimal solution as \( (1,0,0) \) which is quite unrealistic. So, the ranking provided by the proposed similarity measure is more reasonable as the optimal solution considered in this paper is more realistic.

To review the performance of similarity measures let us consider an example. Consider the following four SvNSs

\[
A_1 = \{(x_1,0.1,0.2,0.2), (x_2,0.1,0.2,0.2), (x_3,0.1,0.2,0.2), (x_4,0.1,0.2,0.2)\}
\]

To Table 1 shows the comparison of proposed measures with some existing measures of similarity between two SvNSs.

Table 1 shows the comparison of proposed measures with some existing measures of similarity between two SvNSs.
It is clear from Table 1 that $S_e$ and $S$ are unreasonable in determining similarity between $A_{i1}, A_{i2}$, and $A_{i3}$. Where as $S_{BS1}$, $S_{BS2}$, and $S_{BS3}$ are zero for $A_{i1}$. Thus $S_{BS1}, S_{BS2}, S_{BS3}$ and $S_m$ are able to determine similarity in a better way.

**ONCLUSION**

Distance and similarity measure are significant research area in neutrosophic information theory as they are efficient tools to deal with uncertain and insufficient information. In this paper we have derived averaged distance measure and derived similarity measure using the proposed distance measure and the relation between distance and similarity measure under single valued neutrosophic environment. Further, another similarity measure is proposed for SvNSs. Next, a method is derived to define ideal /optimal point using the existing information as previous literature used $(1,0,0)$ as optimal point which does not exists in reality. Further, proposed similarity measures along with defined optimal point are used to solve multi-attribute decision making problem, which helps in providing rank to all the available alternatives and helps in identifying the best one. An illustrative example is used to demonstrate its application. Finally the proposed similarity measures are compared with some existing similarity measures. Proposed similarity measure $S_m$ is proved to be better than some existing measures.

<table>
<thead>
<tr>
<th>TABLE 1 Comparison between similarity measures</th>
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<tbody>
<tr>
<td>$A_{i1}$</td>
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<tr>
<td>$A_{i2}$</td>
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<td>$A_{i3}$</td>
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**BIBLIOGRAPHY**